Midterm - Analysis of Several Variables (2023-24)
Time: 2.5 hours.
Attempt all questions, giving proper explanations.
You may quote any result proved in class without proof.

1. Give an example of a function $f : \mathbf{R}^2 \to \mathbf{R}$ that is not continuous at the origin but is continuous along every straight line through the origin. **[4 marks]**

2. Let $\mathbf{r} : \mathbf{R}^3 \to \mathbf{R}^3$ be defined as $\mathbf{r}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, and let $r(x, y, z) = \|\mathbf{r}(x, y, z)\|$. Show that $\nabla(r^n) = nr^{n-2}\mathbf{r}$ for integers n. **[3 marks]**

3. Assume $f : \mathbf{R}^n \to \mathbf{R}$ is differentiable at each point of the ball $B(\mathbf{a}; \epsilon)$. If $f'(\mathbf{x}; \mathbf{y}) = 0$ for n independent vectors $\mathbf{y}_1, \cdots, \mathbf{y}_n$ and for every $\mathbf{x}$ in $B(\mathbf{a}; \epsilon)$, prove that f is constant on $B(\mathbf{a}; \epsilon)$. **[4 marks]**

4. Evaluate the directional derivative of $f(x, y, z) = x^2 + y^2 - z^2$ at $(3, 4, 5)$ along the curve of intersection of the two surfaces $2x^2 + 2y^2 - z^2 = 25$ and $x^2 + y^2 = z^2$. **[4 marks]**

5. Find the points on the curve of intersection of the two surfaces

$$x^2 - xy + y^2 - z^2 = 1 \text{ and } x^2 + y^2 = 1$$

which are nearest to the origin. **[4 marks]**

6. Compute $\int_C (x^2 - 2xy)dx + (y^2 - 2xy)dy$, where C is a path from $(-2, 4)$ to $(1, 1)$ along the parabola $y = x^2$. **[4 marks]**

7. Consider $\mathbf{f} : \mathbf{R}^2 \to \mathbf{R}^2$ given by $\mathbf{f}(x, y) = (x, y)$. Let $g(x, y) = \int_{C_1} \mathbf{f} \cdot d\boldsymbol{\alpha} + \int_{C_2} \mathbf{f} \cdot d\boldsymbol{\beta}$ where $\boldsymbol{\alpha}$ is the parametrisation of the straight line segment C_1 from $(0, 0)$ to $(x, 0)$, and $\boldsymbol{\beta}$ is the parametrisation of the straight line segment C_2 from $(x, 0)$ to (x, y). Find the gradient of g. **[4 marks]**

8. State Green's theorem. **[3 marks]**